

Modelling transformations of quadratic functions: A proposal of inductive inquiry

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This paper presents a study about using scientific simulations to enhance the process of mathematical modelling. The main component of the study is a lesson whose major objective is to have students mathematise a trajectory of a projected object and then apply the model to formulate other trajectories by using the properties of function transformations. It was hypothesised that situating the lesson in a modelling environment would enhance the meaning of transformations that are not often conceptualised in mathematics textbooks. The lesson is guided by inductive reasoning. As a medium of data gathering, a free simulation called *Projectile Motion* was used (available at http://phet.colorado.edu/sims/projectile-motion/projectile-motion_en.html).

The inductively organised stages of the activity described in this paper were conducted with a group of ($N = 22$) mathematics students in a high school in Texas. The students' verbal reflections upon this type of novel learning environment supported the study hypothesis. Their perception of the process of studying function transformations has evolved into a meaningful and purposeful experience. Although, the unit was developed for high school math curriculum in the US, its objectives reflect the aims and scope of Australian math curriculum. The Victorian Certificate of Education Study Design (VCAA, 2010) states that students should model investigate and solve problems in unfamiliar situations. The proposed lesson supports this aim.

Adopted conceptual framework

The conceptual framework guiding the instructional unit can be characterised as an inductively organised modelling activity. Mathematical modelling is defined by Winsberg, (2003) as an activity of expressing some aspects of the real world using the language of mathematics and differently by other scholars. According to Bleich, Ledford, Hawley, Polly and Orrill (2006) immersing students in mathematical modelling activities not only expands their views of mathematics but also engages them to delve deeper into mathematics

structures in a search for meanings. While experiencing the processes of mathematical modelling, students are placed in environments in which they actively construct the underlying principles of abstract mathematical knowledge through observations and scientifically derived inferences. In this activity, the scientific inquiry has been supported by four stages of inductive inquiry (Joyce, Weil & Calhoun, 2009) that include focus, conceptual control, inference constituted by a model formulation, and model confirmation. Some modifications were made to highlight the problem statement and data-gathering process. Inductively organised inquiry is also strongly supported by the National Research Council (NRC, 2000) thus reinforcing the idea that reasoning in mathematics classes might benefit students' science knowledge acquisition. It is further hypothesised that by practicing function constructions through the modelling processes, which are often embedded in problem solving, students also improve their problem-solving techniques.

This paper is framed by the above conceptual framework and its goal is to present a lesson for mathematics students to apply the concept of quadratic functions and their transformations to model projectile motion. The lesson is visualised through a dynamic, simulated environment called *Projectile Motion* and created as a part of the PhET Interactive Simulation Project. The choice of using a virtual environment has multiple prompts:

- it is a convenient demonstration in a mathematics classroom typically not designed for real experiments;
- students can be given printed snapshots of the simulated scenarios with which to take data; and
- the mathematical embodiments, for example, the x - y -coordinate axis, are not explicitly drawn and scaled, which facilitates the skills of transferring the tools of mathematics to quantify the variables of real experiments.

The use of a virtual environment to enhance the processes of mathematical modelling is now widely utilised and has been proven to achieve positive learning effects (e.g., Simpson, Hoyles & Noss, 2006).

Lesson conduct

The lesson can be considered one that consolidates the study of quadratic functions and their transformations. If students have not been previously exposed to a simulated learning environment, it is suggested that the teacher take the role of a supporter who will assist in the modelling journey, at least during the initial stages. The simulation can be displayed on a classroom screen and a discussion can take place as students move through the process of function formulation. Alternatively, students can also work individually in a computer lab. It is recommended that the students be given a lab outline with selected snapshots to gather quantifiable data. Auxiliary questions enhancing

other mathematical concepts such as the domain and range can also be embedded. As a means of data gathering, a metric ruler will be used.

Mathematical model formulation

Focus of the activity

The teacher begins the lesson by opening the simulation from http://phet.colorado.edu/sims/projectile-motion/projectile-motion_en.html and demonstrates its features. The simulation can display paths of the projected objects according to prearranged conditions such as initial speed and mass of the object, and the angle of inclination of the cannon. These variables can be entered using the radio buttons on the right side of the simulation. The right side of the menu allows for several objects to be projected—for example, a pumpkin, a piano, or even a human being. This entertaining element engages the student and makes for a friendlier environment. For further analysis, the use of golf ball is recommended; the other mentioned before might distract students' attention.

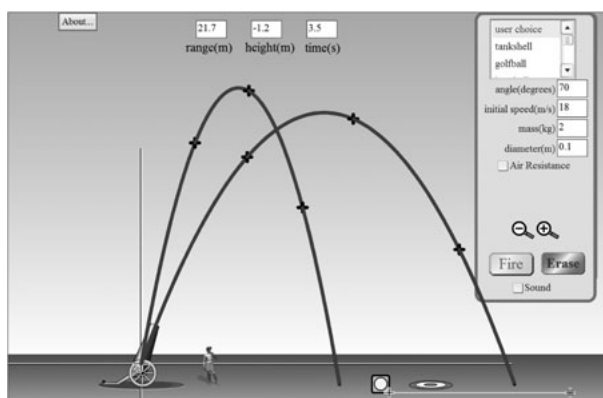


Figure 1. A demonstration of the simulation.

Source: PhET Interactive Simulations, University of Colorado, <http://phet.colorado.edu>.

The teacher demonstrates various trajectories focusing students' attention on the shape of the path of the motion. On the top of the simulation, several physical quantities affecting the motion of the projected objects are displayed:

- range, which in physics means the maximum horizontal distance;
- negative height, which in physics is interpreted as vertical displacement below the initial position; and
- time, which reflects the total time the object is airborne.

Although all of these quantities provide more scientific insight and enhance the physical part of the experiment, they will not be directly used during the lesson. The marks on the paths indicate the distance the object moves after each second of motion, which will also be not used, yet the students might ask the teacher what they represent.

The conceptual control

In order to help students link the context with the lesson objective—to find a mathematical representation of the path of motion—the teacher might posit the following questions:

- How can one describe the path of the projected objects?
- How does the domain and range of the graph change when the angle of inclination of the cannon increases (e.g., assume a constant initial speed)?
- How does the position of the maximum height change if the angle decreases?
- If the speed is constant, what angle produces a maximum horizontal distance of the projectile?
- Does the graph change its general shape under all of these circumstances?

The students realise that the parabolic path of motion is intact under all of these changes. This observation will lead to pattern formulation and mathematisation of the observed path that will be perceived as a graph of a polynomial function.

Problem statement

The teacher explains that the goal of the lesson is to mathematise the paths of the projectiles. He/she simultaneously states the following problem: What type of function can be used to model the path? Having observed the parabolic shape of the graph, the students will suggest a quadratic function to apply. They are then asked what parameters need to be extracted from the scenario to formulate the model. Some students will suggest finding the vertex and using the vertex form, some will refer to an x -intercept form, and some might mention a general standard form of a quadratic function. The teacher might summarise these suggestions as:

- $f(x) = a(x - h)^2 + q$, where $V(h, q)$ represent the coordinates of the vertex;
- $f(x) = a(x - x_1)(x - x_2)$, where x_1, x_2 represent the x -intercepts of the function;
- $f(x) = ax^2 + bx + c$, where a, b and c are the coefficients that can be found by knowing the coordinates of three different points from the parabola and constructing a system of linear equations.

The simulation provides the means to use each of these forms. Consequently, while deriving the function equation for the given graph, it would be up to the students to decide what form to use. As the derived function will serve as a parent function for the next modified scenarios, a vertex form is the recommended application.

Data gathering and inference

The teacher sets the graph to be mathematised by selecting the initial speed, for example, to 18 m/s, and the angle of inclination to 75° . Having decided to use the vertex form, the teacher asks how to quantify the vertex coordinates. The students will experience a dissonance because the x - y -coordinates are not explicitly labelled on the simulation. The first task is then to establish the x - y -coordinates, which might be drawn at the position of the cannon (an image of the axis can be noted on the simulation). After a short discussion, the students realise that the position of the vertex, given by its horizontal and

vertical separation from the origin, can be measured with a ruler. By identifying an additional point from the graph, the parameter a of the parabola can be calculated. There is a virtual ruler embedded in the simulation illustrated by the yellow tape placed along the horizontal and vertical axis shown in Figure 2 and Figure 3 that can be used to demonstrate the process of measuring the coordinates of the vertex.

In order to compute the x -coordinate of the vertex, students can use the properties of symmetry of the parabola and measure the separation between the x -intercepts. By finding the mean value of the separation, that is

$$h = \frac{0 + 16.51}{2} = 8.26 \text{ m}$$

the x -coordinate of the vertex can be concluded. The process of measuring the altitude of the maximum point of the path that is illustrated in Figure 3 returns $q = 15.43 \text{ m}$. Substituting these values to the vertex form results in $f(x)$

$= a(x - 8.26)^2 + 15.43$. In order to complete the process, one more coordinate point from the path is needed. Selecting the point $(0, 0)$ produces $0 = a(0 - 8.26)^2 + 15.43$. Solving the equation for $a = -0.23$, thus the complete form of the mathematical representation of the path, is $f(x) = -0.23(x - 8.26)^2 + 15.43$.

Confirmation

In the process of modelling, verification and confirmation of the derived model play a pivotal role. The verification process also has a profound effect on students' self-esteem. It will either verify or not that they learned the necessary skills to perform the tasks of mathematical modelling. The teacher can ask the students for the possible means of the model verification. Some students will suggest using a graphing calculator, which is reasonable. Observing the graph reassembling the given projectile assures the proficiency of their skills. If a graphing technology is not available, the teacher might suggest verifying a selected path coordinate by the following method: select a point from the path, measure its horizontal distance, and calculate its respective vertical position using the derived function. Measuring the vertical distance with a ruler and comparing its magnitude with the calculated one will also validate

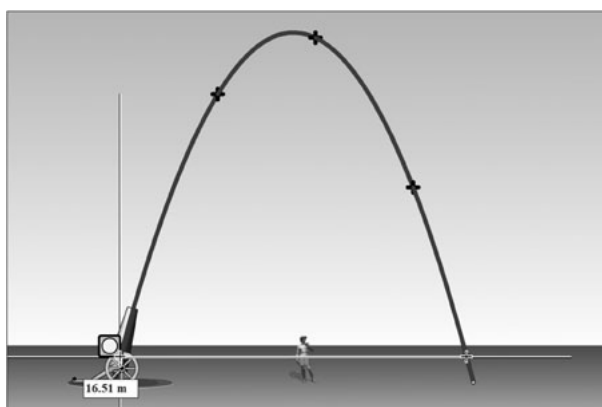


Figure 2. Measuring the horizontal position of the vertex by taking the mean of the x -intercepts. Source: PhET Interactive Simulations, University of Colorado, <http://phet.colorado.edu>.

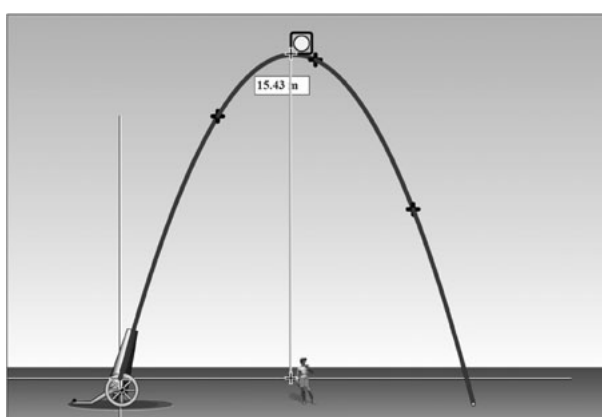


Figure 3. Measuring the y -coordinate of the vertex. Source: PhET Interactive Simulations, University of Colorado, <http://phet.colorado.edu>.

the model. Once the model is verified, it can be further used to mathematise other paths by the means of transformations.

Further model validations by transformations

The purpose of this section is to highlight the relevance of using the concept of function transformations. This section can also be perceived as a further confirmation of the model. The transformations will be narrowed to vertical and horizontal shifts as well as the reflection along the vertical axis. The simulation also allows for other transformation investigations that can be assigned as independent student work. These will be suggested in the *extensions* section of this article. While working on the transformations, the initial speed and mass of the projected object must remain unchanged with respect to the general function derived during the first part of the lesson. The physical parameters that will be modified are the position of the cannon and its angle of inclination.

Vertical transformation

The support of the cannon can be vertically extended by placing the cursor on it and dragging it upwards. For the vertical transformation to be applied, the frame of reference must be left intact and thus it must stay on the ground. The following snapshot illustrates the original path and the path vertically transformed. While preparing the snapshots for student use, the teacher might highlight the frame position of the x - y -axis of the *parent* graph as a reference for inducing transformations.

Some auxiliary questions that might be addressed here are how the position of the vertex of the parabola changed and consequently how the domain and the range of the function changed. The students realise that the vertex moved upward and the altitude of the cannon can be used to quantify the vertical transformation of the path that is illustrated in Figure 4. The students will use

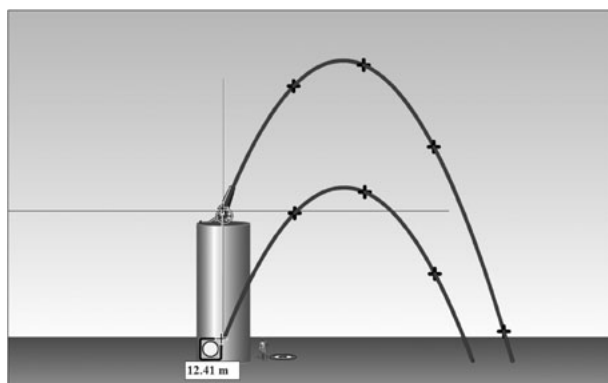


Figure 4. Modelling the vertical transformation.
(Source: PhET Interactive Simulations, University of Colorado,
<http://phet.colorado.edu>.)

their rulers to measure the magnitude of the vertical transformation. The final form of the function, using the magnitude of the transformation from Figure 4, is $f(x) = -0.23(x - 8.26)^2 + 27.84$ where the position of the vertex is expressed in metres.

Horizontal transformation

In a similar fashion, a horizontal transformation can be applied. The previous graph remains on the simulation to serve as a reference. In this case, the cannon is moved to the left. The teacher might ask the students to predict if the parameter a , b or c changed with reference to the parent function. The tape measurement illustrated in Figure 5 represents the magnitude of the horizontal transformation. The students realise that adding the magnitude of the transformation to the independent variable of the function will generate the new function, which will materialise as $f(x) = -0.23(x - 1.52)^2 + 27.84$.

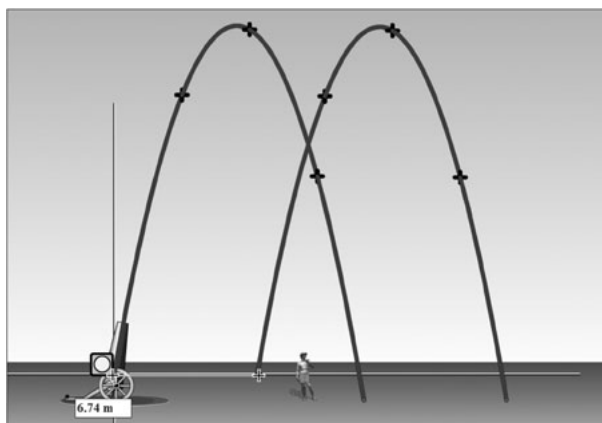


Figure 5. Modelling the horizontal transformation.
(Source: PhET Interactive Simulations, University of Colorado,
<http://phet.colorado.edu>.)

Reflection of the y-axis

An interesting case is presented by increasing the angle of inclination of the cannon to 100 degrees. The students observe that this modification changed the horizontal component of the object velocity from a positive to a negative value thus the projected object now moves in a negative direction. The students can be asked if, in this case, the parameter a of the parabola changed. This question usually puzzles the students, because the cannon position had not changed yet the trajectory is now different. If this happens, some auxiliary questions targeting the properties of the parameter a are suggested. The students will realise that the parabola did not stretch nor compress, which implies that the parameter a did not change. The new graph can be generated by producing a reflection of the existing graph of the y-axis, thus $f(x) = -0.23(-x - 1.52)^2 + 27.84$ will resemble the new graph illustrated on the left side in Figure 6.

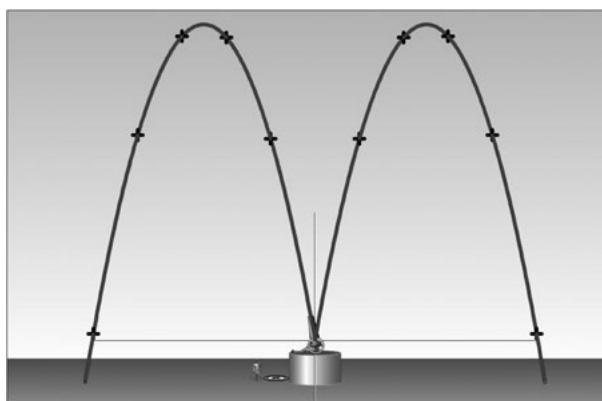


Figure 6. Modelling the vertical reflection.
(Source: PhET Interactive Simulations, University of Colorado,
<http://phet.colorado.edu>.)

Extensions

The following are some special cases that might be included in independent student work. A discussion by the teacher on conceptualizing the various scenarios is recommended. Although the task is to mathematise the path of motion, the idea of transformations might not be used in all cases.

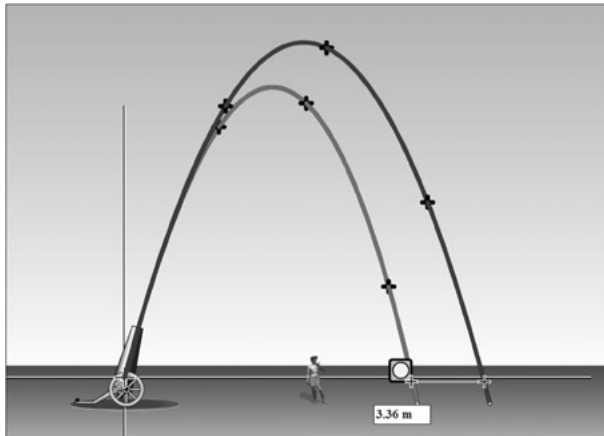


Figure 7. Modelling the vertical and horizontal compression.
(Source: PhET Interactive Simulations, University of Colorado,
<http://phet.colorado.edu>.)

Effect of air resistance

This simulation provides a great opportunity to investigate the change in the parabola's shape when the force of air resistance acts on the projected objects. As illustrated in Figure 7, when the force of air resistance is present, there is an additional decrease of the objects' speed that manifests as a compression of the path of motion. The inner (diminished) graph in Figure 7 depicts the effects of air resistance. The students may

be asked to conclude the type of transformations of the larger (original) graph) that will result in generating the diminished one. Some students will choose to measure the position of the vertex and proceed as in the parent-function formulation. Contrasting both functions and identifying ratios of compressions might also lead students to the correct conclusions.

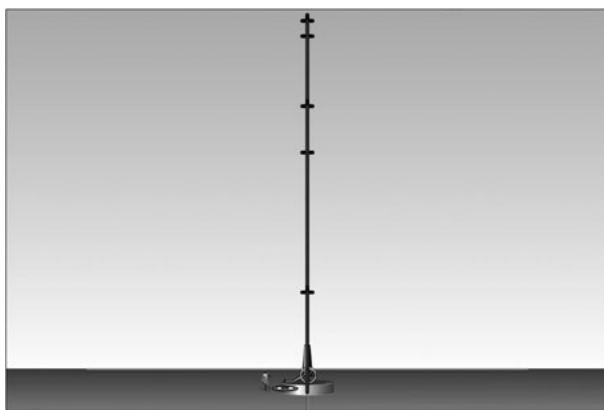


Figure 8. Modelling the vertical motion. (Source: PhET
Interactive Simulations, University of Colorado, <http://phet.colorado.edu>.)

Analysis of free fall (projectile motion with an inclination of 90°)

When shown this scenario students experience a dissonance as the vertical path of motion does not represent a function. Some of them will suggest that $x = 0$ should be used yet the teacher suggests imposing some restrictions on the range of the relation. The restriction can be quantified by using a ruler and measuring the length of the vertical segment.

Projectile motion with an inclination of 0°

An equally interesting case is one in which the parabola displays only one of its parts as illustrated in Figure 9. Students might be directed to identify the position of the vertex of the parabola, which can be established at the location of the cannon. The change of the shape of the parabola is attributed

to a change of the initial speed of the projectile. The students will observe that an increase of magnitude of the horizontal velocity results in an increase of the horizontal extension of the graph; that is, mathematised by a decrease of the magnitude of the leading term a of the corresponding quadratic function.

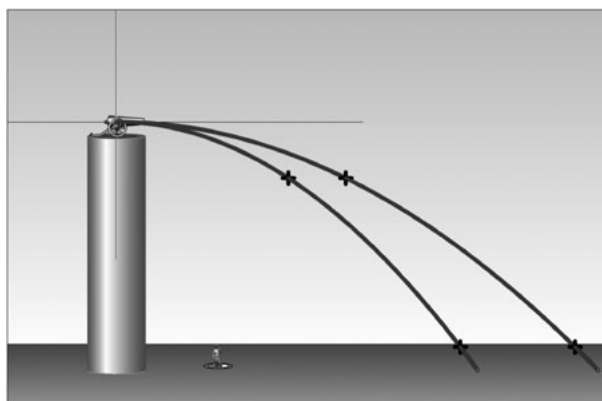


Figure 9. Modelling the vertical stretch. (Source: PhET Interactive Simulations, University of Colorado, <http://phet.colorado.edu>.)

Student opinions and general conclusions

To evaluate student perceptions and learning effects of being immersed in this interactive learning environment, two reflective-type questions were asked after the lesson.

1. Did the simulated environment help you with understanding the process of applying function transformations?
2. What stages of the lesson did you find the most effective?

The purpose of first question was to learn if students could make transitions from observing the simulated environment to being able to model it using the tools of algebra.

Almost all students ($N = 21$, 95%) felt that the lesson improved their understanding of function transformations. In supporting the claim, most of them highlighted the visual and dynamic aspects of the teaching aid applied. For example, some said that “Looking at pictures and examples helped to put things into a clearer perspective,” or, “The simulations allowed for a better mental picture”. Since many of these students were concurrently taking a physics course, the lesson supported an integrated view of science learning: “There were many examples and it tied in what we already learned in physics”; “It showed real work examples applied to physics”. Others focussed on evaluating overall quality of the simulations: “It provided a visual and was somewhat simple”; “It helped me think about what I needed to do to find the function, which I have always had trouble with”.

The purpose of the second question was to learn more about the effects of various processes applied during the functions formulations. Since enacting the parent function was a pivotal point of most of the scenarios, some students realised the importance of correctly recognizing the function and building the transformed once using it: “I learned that the main equation never changes when the graph is transformed”; “I learned that even though the graphs looked different, the parent function could be the same”. Some students appreciated the mode of extracting necessary data and the graph production: “I learned how to take measurements to make up a function”; “I learned what

physical changes affect the graph”; “I liked the changes to the graph after the angle of inclination changed”; “I liked the example of horizontal movement of the parabola in real terms”. Others were more general with their thoughts: “I learned what the parameters a , b and c tell you about the transformations”; “Transformations are much simpler, than I once thought”.

In sum, it seems that conducting the modelling lesson showed students a different face of mathematics concepts. From their responses, one can conclude that abstract mathematical ideas presented in physical embodiments become tangible and more meaningful to them simultaneously showing them their importance to understand science.

Although this lesson utilised only one simulation, other simulations designed by PhET also provide great contexts for modelling mathematics activities. *Wave on String* discussed in Sokolowski and Rackley (2011), for example, supports applications of trigonometric functions; *Balancing Act* proposes applying an interactive lever arm to introduce the process of building and solving linear equations (Sokolowski, 2013).

This case study was undertaken to support the notion that modelling activities are beneficial to students because they present math structures in ways that resonate with their prior knowledge and experiences. It is further hypothesised that being able to induce the tools of mathematics to quantify scientific experimentations enhances students’ problem-solving techniques, which materialises as a valuable undertaking for further studies.

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